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Abstract

We find necessary and sufficient conditions for the stability of all work-conserving policies for multiclass fluid queueing networks with two stations. Furthermore, we find new sufficient conditions for the stability of multiclass queueing networks involving any number of stations and conjecture that these conditions are also necessary. Previous research had identified sufficient conditions through the use of a particular class (monotone piecewise linear convex) potential functions. We show that for two-station systems it is not possible for this class of potential function to give the new (sharp) conditions.

1 Introduction

The problem of establishing conditions under which a multiclass queueing network is stable under a particular policy has attracted a lot of attention in recent years. It is known that for single class (Borovkov [1], Sigman [16], Meyn and Down [14]) and multiclass acyclic queueing networks a necessary and sufficient condition for stability of all work-conserving policies is

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that the traffic intensity at each station of the network is less than one. For multiclass networks with feedback, Kumar and Seidman [11] (see also Lu and Kumar [12] and Rybko and Stolyar [15]) have identified particular priority policies that lead to instability even if the traffic intensity at each station of the network is less than one. More surprisingly, Bramson [2] has shown that these instability phenomena are present even for the standard FIFO policy. It is therefore, a rather interesting problem to identify the right set of necessary and sufficient conditions for stability of multiclass queueing networks.

In recent years researchers have identified progressively sharper sufficient conditions for stability of all work-conserving policies through the use of Lyapunov functions. Kumar and Meyn [10] used quadratic potential functions, while Botvich and Zamyatin [3], Dai and Weiss [7], and Down and Meyn [8] used piecewise linear convex potential functions. In all cases, it was established that a multiclass network is stable if certain linear programming problems are bounded. To the best of our knowledge the sharpest such conditions are those of [7] and [8] obtained through the use of piecewise linear convex potential functions. For some specific examples (for example in [3]), the conditions obtained are indeed sharp. In general, however, the problem of establishing the exact stability region, i.e., sharp necessary and sufficient conditions for stability, is open. Furthermore, it is not known whether the potential function method with piecewise linear convex functions (or with any convex potential function) has the power of establishing the exact stability region. Finally, Chen and Zhang [5] have found some sufficient (but not necessary) conditions for the stability of multiclass queueing networks under FIFO.

Dai [6] and Meyn [13] have shown that a stochastic multiclass network is stable if and only if the associated fluid limit (a deterministic network) is stable. For this reason, while this paper concentrates on deterministic fluid models, there are immediate ramifications of our results for the case of stochastic models.

The contributions of this paper can be summarized as follows:

1. We find, in Section 3, the exact stability region for two-station multiclass networks by a method that looks at the detailed structure of possible trajectories. The stability

condition is expressed in terms of a linear program.

2. We find, in Section 4, new sufficient conditions for multiclass networks with more than two stations that we believe to be necessary, although we were unable to establish necessity. The conditions are again expressed in terms of a linear program. Unfortunately, the number of variables involved increases exponentially with the number of stations, but we believe that this is unavoidable.

3. We fully characterize, in Section 5, the power of the potential function method based on piecewise linear monotone convex functions, for the two–station case. In particular, we show that one never need consider potential functions involving more than two linear pieces. We also derive a linear program that searches for such potential functions. We further show that this class of potential functions cannot find the exact stability region, thus establishing certain intrinsic limitations of earlier approaches.

2 Notation

We introduce a fluid model (α, μ, P, C) consisting of n classes $C_1, ..., C_n$ and J service stations $1, \ldots, J$ as follows. Each class is served at a particular station. Let σ_j be the set of classes that are served in station j. The external arrival rate for class i is α_i and the service rate is μ_i. Let $\alpha = (\alpha_1, ...\alpha_n)'$ and $\mu = (\mu_1, ..., \mu_n)'$. After service completion a fraction p_{ij} of class i customers becomes of class j and a fraction $1 - \sum_j p_{ij}$ exits the system. Let P be the substochastic matrix $P = (P_{ij})_{1 \leq i,j \leq n}$. Finally, we define the $J \times n$ matrix C as follows: $c_{jk} = 1$ if class k is served at station j and $c_{jk} = 0$ otherwise. We let $M = \text{diag}\{\mu_1, ..., \mu_n\}$ and assume that the matrix P has spectral radius less that one.

Any scheduling policy can be described in terms of the variables $T_k(t)$ defined as the amount of time class k is being served in the interval $[0, t]$, and $Q_k(t)$ defined as the queue length for class k at time t. We let $T(t) = (T_1(t), \ldots, T_n(t))'$ and $Q(t) = (Q_1(t), \ldots, Q_n(t))'$. Throughout the paper we call $Q(t)$ the trajectory of the fluid process under the allocation process $T(t)$. Given the initial condition $Q(0)$, the dynamics of the queue length process are as follows:

$$Q_k(t) = Q_k(0) + \alpha_k t + \sum_{i=1}^{n} \mu_i T_i(t) p_{ik} - \mu_k T_k(t) \geq 0, \quad k = 1, \dots, n,$$

or in matrix form:

$$Q(t) = Q(0) + \alpha t + [P' - I] M T(t) \geq 0.$$

We assume that the allocation process satisfies the following conditions:

1. $T(0) = 0$,

2.(Feasibility) For any $t_2 > t_1 \geq 0$ and any station i:

$$\sum_{k \in \sigma_i} [T_k(t_2) - T_k(t_1)] \leq t_2 - t_1, \tag{1}$$

and $T_k(t)$ is nondecreasing.

3. (Work-conservation) If for all $t \in [t_1, t_2]$ we have $\sum_{k \in \sigma_i} Q_k(t) > 0$ for some station i, then

$$\sum_{k \in \sigma_i} [T_k(t_2) - T_k(t_1)] = t_2 - t_1. \tag{2}$$

Any scheduling policy satisfying all the above properties is called a (feasible) work-conserving policy.

An alternative characterization of the above requirements is to introduce for any station i, the cumulative idling process:

$$U_i(t) = t - \sum_{k \in \sigma_i} T_k(t).$$

The feasibility condition (1) then requires that $U_i(t)$ be nonnegative and nondecreasing, while the work-conservation condition is rewritten as follows: if for all $t \in [t_1, t_2]$ we have $\sum_{k \in \sigma_i} Q_k(t) > 0$, then

$$U_i(t_1) = U_i(t_2). \tag{3}$$

Following Chen [4], a fluid network (α, μ, P, C) is said to be stable for all work-conserving policies if for every work-conserving allocation process $T(t)$ and every initial condition $Q(0)$, there exists a finite time t_0 such that $Q(t_0) = 0$.

A necessary condition for stability (see Chen [4]) is that the traffic intensity vector ρ defined by $\rho = CM^{-1}[I - P']^{-1}\alpha$, satisfies

$$\rho < e, \tag{4}$$

where $e = (1, \dots, 1)'$. As mentioned in the introduction, for general multiclass networks with feedback, this condition is not sufficient. Our goal in the next section is to establish necessary and sufficient conditions for the stability of a multiclass fluid network with two stations, given that $\rho < e$. In preparation for this analysis, we introduce some useful notation.

We refer to $Q(t) \in R_+^n$ as the state of the system at time $t \geq 0$. We partition the set $R_+^n - \{0\}$ of nonzero states into the following finite family of subspaces. For any non-empty set of service stations $S \subset \{1, 2, ..., J\}$, we let

$$R_S = \{x \in R_+^n : \forall i \in S, \ \sum_{k \in \sigma_i} x_k > 0, \ \text{and} \ \forall i \notin S, \ \sum_{k \in \sigma_i} x_k = 0\},$$

i.e., R_S corresponds to states for which all stations in S are busy, while all other stations have empty buffers.

3 Stability conditions for multiclass two-station fluid networks

In this section we establish necessary and sufficient conditions for stability, for the case where $J = 2$, i.e., for multiclass networks with two stations. Throughout this section, we assume that $\rho < e$ because otherwise the stability problem is trivial.

We denote by R_1, R_2 and R_{12} the subspaces corresponding to $S = \{1\}, \{2\}, \{1, 2\}$, respectively, as defined at the end of Section 2. In particular, for $Q \in R_1$ station 2 has no customers, for $Q \in R_2$ station 1 has no customers, while for $Q \in R_{12}$ both stations have customers in queue. The proposition that follows states that a trajectory can be broken down into subtrajectories of four different types.

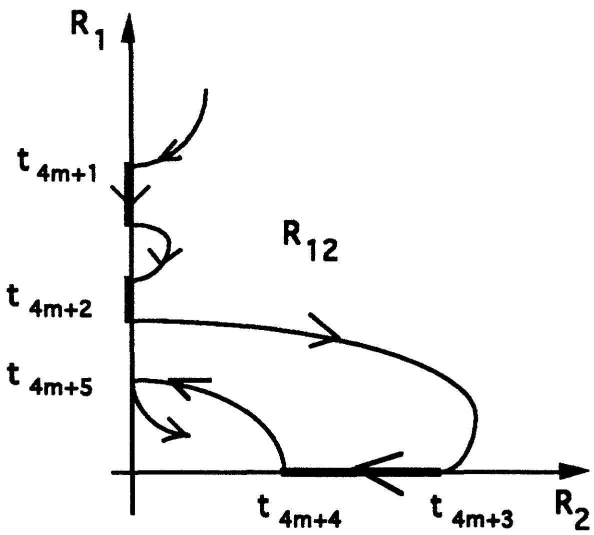

Figure 1: The times t_i for a typical trajectory.

Proposition 1 *Consider a stable work-conserving trajectory $Q(t)$ and let T be the smallest time such that $Q(T) = 0$. There exists a (finite or infinite) nondecreasing sequence t_i such that $\sup_i t_i = T$ and such that for all times less than T the following hold:*

$Q(t_{4m+1}) \in R_1$ *and for* $t \in [t_{4m+1}, t_{4m+2}]$, $Q(t) \in R_1 \cup R_{12}$;

$Q(t_{4m+2}) \in R_1$ *and for* $t \in (t_{4m+2}, t_{4m+3})$, $Q(t) \in R_{12}$;

$Q(t_{4m+3}) \in R_2$ *and for* $t \in [t_{4m+3}, t_{4m+4}]$, $Q(t) \in R_2 \cup R_{12}$;

$Q(t_{4m+4}) \in R_2$ *and for* $t \in (t_{4m+4}, t_{4m+5})$, $Q(t) \in R_{12}$.

Proof: This is a simple consequence of the fact that starting in R_1, the system can get to R_2 only by first going through R_{12}, and vice versa; see Figure 1. In particular, once t_{4m+1} has been defined, we may let $t_{4m+3} = \min\{t > t_{4m+1} \mid Q(t) \in R_2\}$ and $t_{4m+2} = \max\{t < t_{4m+3} \mid Q(t) \in R_1\}$. [In case $Q(t)$ never enters R_2 after time t_{4m+1}, then the preceding definition of t_{4m+3} is inapplicable; however, in this case, the system gets to $Q(T) = 0$ without ever leaving $R_1 \cup R_{12}$. Thus, $[t_{4m+1}, T)$ can be taken as the last interval.] Having thus defined t_{4m+3}, the times t_{4m+4} and t_{4m+5} are defined similarly. □

3.1 Bounds for the strong busy period of stable work-conserving policies

In this subsection we find an upper bound on the time that stable work-conserving policies take to empty the fluid network starting with an initial condition $Q(0)$. This time is usually called the strong busy period. This result is of independent interest, as it contributes to our understanding of the performance of the network; it is also the key to our stability analysis in the next subsection.

Proposition 2 *Consider a stable work-conserving policy $T(t)$ starting with initial condition $Q(0) \neq 0$. Let T be the smallest time such that $Q(T) = 0$. Then; T is bounded above by the optimal value of the following linear program to be called $LP[Q(0)]$:*

$$\text{maximize } \sum_{j=1}^{4} T_j$$

subject to

$$T_1 = \sum_{k\in\sigma_1} T_k^1, \quad T_1 \geq \sum_{k\in\sigma_2} T_k^1,$$

$$T_2 = \sum_{k\in\sigma_1} T_k^2, \quad T_2 = \sum_{k\in\sigma_2} T_k^2,$$

$$T_3 \geq \sum_{k\in\sigma_1} T_k^3, \quad T_3 = \sum_{k\in\sigma_2} T_k^3,$$

$$T_4 = \sum_{k\in\sigma_1} T_k^4, \quad T_4 = \sum_{k\in\sigma_2} T_k^4,$$

$\forall\, k \in \sigma_2$:

$$\alpha_k T_1 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^1 - \mu_k T_k^1 = 0,$$

$$\alpha_k T_2 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^2 - \mu_k T_k^2 \geq 0,$$

$$\alpha_k T_4 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^4 - \mu_k T_k^4 \leq 0,$$

$\forall\, k \in \sigma_1$:

$$\alpha_k T_3 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^3 - \mu_k T_k^3 = 0,$$

$$\alpha_k T_4 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^4 - \mu_k T_k^4 \geq 0,$$

$$\alpha_k T_2 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^2 - \mu_k T_k^2 \leq 0,$$

$\forall k \in \{1, \ldots, n\}$:

$$\alpha_k \sum_{j=1}^{4} T_j + \sum_{i=1}^{n} \mu_i p_{ik} \sum_{j=1}^{4} T_i^j - \mu_k \sum_{j=1}^{4} T_k^j = -Q_k(0), \tag{5}$$

$$T_j \geq 0, \quad T_k^j \geq 0.$$

Proof: Consider a stable work conserving policy with initial condition $Q(0) \neq 0$. Without loss of generality, we only provide the proof for the case $Q(0) \in R_1$; the proof for the other cases is essentially identical. Let $t_1 = 0$ and let the times t_j be as in the statement of Proposition 1. For $j = 1, \ldots, 4$ we introduce the following variables:

$$T_j = \sum_{m=0}^{\infty} (t_{4m+j+1} - t_{4m+j}) \tag{6}$$

and

$$T_k^j = \sum_{m=0}^{\infty} (T_k(t_{4m+j+1}) - T_k(t_{4m+j})). \tag{7}$$

Intuitively, T_1 is the total amount of time the trajectory spends in R_1 as well as in excursions from R_1 into R_{12} and back into R_1; T_2 is the total amount of time the trajectory spends in R_{12} coming from R_1 and going to R_2; T_3 is the total amount of time the trajectory spends in R_2 as well as in excursions from R_2 into R_{12} and back into R_2; finally, T_4 is the total amount of time the trajectory spends in R_{12} coming from R_2 and going to R_1. Clearly $T_j \geq 0$ and the first time that $Q(t)$ becomes zero is given by $T = T_1 + T_2 + T_3 + T_4$. Note that for every class k, T_k^1, T_k^2, T_k^3 and T_k^4 is the total work allocated to class k, during the time intervals that enter in the definitions of T_1, T_2, T_3, T_4, respectively.

For all $t \in [t_{4m+1}, t_{4m+2}]$, we have $Q(t) \in R_1 \cup R_{12}$, and therefore $\sum_{k \in \sigma_1} Q_k(t) > 0$. Because the policy is work-conserving,

$$t_{4m+2} - t_{4m+1} = \sum_{k \in \sigma_1} (T_k(t_{4m+2}) - T_k(t_{4m+1})). \tag{8}$$

By summing over $m \geq 0$ we obtain that

$$T_1 = \sum_{k \in \sigma_1} T_k^1,$$

which simply expresses the work conservation in station 1, while the trajectory is in $R_1 \cup R_{12}$ (station 1 busy). Similarly, work conservation for station 2, while the trajectory is in $R_2 \cup R_{12}$ (station 2 busy) leads to

$$T_3 = \sum_{k \in \sigma_2} T_k^3.$$

Moreover, for $t \in (t_{4m+2}, t_{4m+3}) \cup (t_{4m+4}, t_{4m+5})$, we have $Q(t) \in R_{12}$, and work conservation for both stations leads to

$$T_2 = \sum_{k \in \sigma_1} T_k^2 = \sum_{k \in \sigma_2} T_k^2, \qquad T_4 = \sum_{k \in \sigma_1} T_k^4 = \sum_{k \in \sigma_2} T_k^4.$$

For every station j, we have

$$\sum_{k \in \sigma_j} (T_k(t_{i+1}) - T_k(t_i)) \leq t_{i+1} - t_i,$$

leading to

$$T_1 \geq \sum_{k \in \sigma_2} T_k^1, \;\; T_3 \geq \sum_{k \in \sigma_1} T_k^3.$$

By definition of the times t_i, we have $Q(t_{4m+1}) \in R_1$ and $Q(t_{4m+2}) \in R_1$. Thus, for all $k \in \sigma_2$ we have

$$Q_k(t_{4m+1}) = Q_k(t_{4m+2}) = 0,$$

which leads to

$$\alpha_k(t_{4m+2} - t_{4m+1}) + \sum_{i=1}^{n} \mu_i p_{ik}(T_i(t_{4m+2}) - T_i(t_{4m+1})) - \mu_k(T_k(t_{4m+2}) - T_k(t_{4m+1})) = 0, \; k \in \sigma_2.$$

Summing over all $m \geq 0$, we obtain

$$\alpha_k T_1 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^1 - \mu_k T_k^1 = 0, \; k \in \sigma_2.$$

Similarly, for $k \in \sigma_1$, we have $Q_k(t_{4m+3}) = Q_k(t_{4m+4}) = 0$, which yields

$$\alpha_k(t_{4m+4} - t_{4m+3}) + \sum_{i=1}^{n} \mu_i p_{ik}(T_i(t_{4m+4}) - T_i(t_{4m+3})) - \mu_k(T_k(t_{4m+4}) - T_k(t_{4m+3})) = 0, \; k \in \sigma_1,$$

and leads to

$$\alpha_k T_3 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^3 - \mu_k T_k^3 = 0, \ k \in \sigma_1.$$

Since $Q(t_{4m+2}) \in R_1$ and $Q(t_{4m+3}) \in R_2$, we obtain

$$0 = \sum_{k \in \sigma_2} Q_k(t_{4m+2}) < \sum_{k \in \sigma_2} Q_k(t_{4m+3})$$

and

$$0 = \sum_{k \in \sigma_1} Q_k(t_{4m+3}) < \sum_{k \in \sigma_1} Q_k(t_{4m+2}),$$

which implies that for all $k \in \sigma_2$, $Q_k(t_{4m+3}) - Q_k(t_{4m+2}) \geq 0$, leading to

$$\alpha_k(t_{4m+3} - t_{4m+2}) + \sum_{i=1}^{n} \mu_i p_{ik}(T_i(t_{4m+3}) - T_i(t_{4m+2})) - \mu_k(T_k(t_{4m+3}) - T_k(t_{4m+2})) \geq 0, \ k \in \sigma_2.$$

Summing over all $m \geq 0$, we obtain

$$\alpha_k T_2 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^2 - \mu_k T_k^2 \geq 0, \ k \in \sigma_2.$$

Similarly, for all $k \in \sigma_1$, $Q_k(t_{4m+3}) - Q_k(t_{4m+2}) \leq 0$, leading to

$$\alpha_k(t_{4m+3} - t_{4m+2}) + \sum_{i=1}^{n} \mu_i p_{ik}(T_i(t_{4m+3}) - T_i(t_{4m+2})) - \mu_k(T_k(t_{4m+3}) - T_k(t_{4m+2})) \leq 0, \ k \in \sigma_1,$$

and therefore,

$$\alpha_k T_2 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^2 - \mu_k T_k^2 \leq 0, \ k \in \sigma_1.$$

Finally, since $Q(t_{4m+4}) \in R_2$ and $Q(t_{4m+5}) \in R_1$, we obtain:

$$\alpha_k(t_{4m+5} - t_{4m+4}) + \sum_{i=1}^{n} \mu_i p_{ik}(T_i(t_{4m+5}) - T_i(t_{4m+4})) - \mu_k(T_k(t_{4m+5}) - T_k(t_{4m+4})) \geq 0, \ k \in \sigma_1,$$

$$\alpha_k(t_{4m+5} - t_{4m+4}) + \sum_{i=1}^{n} \mu_i p_{ik}(T_i(t_{4m+5}) - T_i(t_{4m+4})) - \mu_k(T_k(t_{4m+5}) - T_k(t_{4m+4})) \leq 0, \ k \in \sigma_2,$$

leading respectively to

$$\alpha_k T_4 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^4 - \mu_k T_k^4 \geq 0, \ k \in \sigma_1,$$

$$\alpha_k T_4 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^4 - \mu_k T_k^4 \leq 0, \; k \in \sigma_2.$$

Recall that $T = \sum_{j=1}^{4} T_j$. Then, from the dynamics of the network

$$Q_k(T) = Q_k(0) + \alpha_k T + \sum_{i=1}^{n} \mu_i p_{ik} \sum_{j=1}^{4} T_i^j - \mu_k \sum_{j=1}^{4} T_k^j.$$

Since $Q(T) = 0$, we obtain

$$\alpha_k T + \sum_{i=1}^{n} \mu_i p_{ik} \sum_{j=1}^{4} T_i^j - \mu_k \sum_{j=1}^{4} T_k^j = -Q_k(0), \qquad k = 1, \ldots, n.$$

We have shown that all of the constraints of the linear program $LP[Q(0)]$ must be satisfied. It follows that T must be bounded above by the value of that linear program. □

The linear program LP[Q(0)] gives an upper bound on the strong busy period of all stable work-conserving policies. Similarly, if we minimize $\sum_{i=1}^{4} T_i$ we find a lower bound on the time it takes for the network to empty using a work-conserving policy starting from an initial condition $Q(0)$. The lower bound is particularly interesting as it gives information on the best possible performance.

3.2 Sufficient conditions for stability

In this subsection, we derive sufficient conditions for stability of the fluid network. These sufficient conditions involve the linear program LP[0] which is defined exactly as the linear program LP[Q(0)] of the preceding subsection, except that the right–hand side variables $Q_k(0)$ in the constraints (5) are set to zero.

Theorem 1 (Sufficient Conditions for stability) *Consider the following set of linear inequalities in* $4(n+1)$ *variables*

$$T_1 = \sum_{k \in \sigma_1} T_k^1, \;\; T_1 \geq \sum_{k \in \sigma_2} T_k^1, \tag{9}$$

$$T_2 = \sum_{k \in \sigma_1} T_k^2, \;\; T_2 = \sum_{k \in \sigma_2} T_k^2, \tag{10}$$

$$T_3 \geq \sum_{k\in\sigma_1} T_k^3, \quad T_3 = \sum_{k\in\sigma_2} T_k^3, \tag{11}$$

$$T_4 = \sum_{k\in\sigma_1} T_k^4, \quad T_4 = \sum_{k\in\sigma_2} T_k^4, \tag{12}$$

$\forall\, k \in \sigma_2$:

$$\alpha_k T_1 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^1 - \mu_k T_k^1 = 0, \tag{13}$$

$$\alpha_k T_2 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^2 - \mu_k T_k^2 \geq 0, \tag{14}$$

$$\alpha_k T_4 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^4 - \mu_k T_k^4 \leq 0, \tag{15}$$

$\forall\, k \in \sigma_1$:

$$\alpha_k T_3 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^3 - \mu_k T_k^3 = 0, \tag{16}$$

$$\alpha_k T_4 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^4 - \mu_k T_k^4 \geq 0, \tag{17}$$

$$\alpha_k T_2 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^2 - \mu_k T_k^2 \leq 0, \tag{18}$$

$\forall k \in \{1, \ldots, n\}$:

$$\alpha_k \sum_{j=1}^{4} T_j + \sum_{i=1}^{n} \mu_i p_{ik} \sum_{j=1}^{4} T_i^j - \mu_k \sum_{j=1}^{4} T_k^j = 0, \tag{19}$$

$$T_j \geq 0, \quad T_k^j \geq 0,$$

to be referred to as $LP[0]$. If $LP[0]$ has has zero as the only feasible solution, then the multiclass fluid network (α, μ, P, C) is stable for all work-conserving policies.

Proof: Let us assume that zero is the only feasible solution of $LP[0]$. Let us also assume that there exists an initial condition $Q(0) \neq 0$ and a work-conserving policy such that $Q(t)$ never becomes zero. We will derive a contradiction.

Recall that the constraints in $LP[0]$ and in $LP[Q(0)]$ are the same except that the right hand-side in (5) is changed from $-Q_k(0)$ to zero. Using linear programming theory and since 0 is the only feasible solution of $LP[0]$, it follows that the feasible set of $LP[Q(0)]$ is bounded. Let Z be the optimal value of the objective function in $LP[Q(0)]$, which is finite.

Let us now consider the unstable policy starting from $Q(0)$. Let us follow this policy up to time Z; from then on, let us switch to some stable work-conserving policy (under our standing assumption that $\rho < e$, it is known that such a policy exists.) We then obtain a work-conserving policy that, starting from $Q(0)$, eventually leads the state to zero, say at some time T. By construction $T > Z$. On the other hand, Proposition 2 asserts that $T \leq Z$. This is a contradiction and the proof is complete. □

3.3 Necessary conditions for stability

In this section we show that the conditions of Theorem 1 are also necessary. In particular, we show that if the linear program $LP[0]$ has a nonzero solution (T_j, T_k^j), $j = 1, \ldots, 4$, $k = 1, \ldots, n$, then there exists a work-conserving policy and an initial condition $Q(0) \neq 0$, such that for some time $\tau > 0$, $Q(\tau) = Q(0)$. By repeating the same policy each time that the state $Q(0)$ is revisited, the system never empties and therefore the fluid network is unstable. In preparation of the instability theorem we prove the following proposition.

Proposition 3 *If (T_j, T_k^j), $j = 1, \ldots, 4$, $k = 1, \ldots, n$, is a nonzero solution of $LP[0]$, then $T_j > 0$ for all $j = 1, \ldots, 4$.*

Proof

Suppose $T_1 = 0$. Then from (9) $T_k^1 = 0$ for all $k = 1, \ldots, n$ and therefore, from (19) we obtain for all $k = 1, \ldots, n$,

$$\alpha_k(T_2 + T_3 + T_4) + \sum_{i=1}^{n} \mu_i p_{ik}(T_i^2 + T_i^3 + T_i^4) - \mu_k(T_k^2 + T_k^3 + T_k^4) = 0$$

or in matrix form, with $T^j = (T_1^j, \ldots, T_n^j)'$,

$$\alpha(T_2 + T_3 + T_4) + [P' - I]M[T^2 + T^3 + T^4] = 0$$

Multiplying both sides from the left by $CM^{-1}[I - P']^{-1}$ we obtain

$$\begin{pmatrix} \rho_1 - 1 \\ \rho_2 - 1 \end{pmatrix} (T_2 + T_3 + T_4) + \begin{pmatrix} T_2 + T_3 + T_4 - \sum_{k \in \sigma_1}(T_k^2 + T_k^3 + T_k^4) \\ T_2 + T_3 + T_4 - \sum_{k \in \sigma_2}(T_k^2 + T_k^3 + T_k^4) \end{pmatrix} = 0.$$

But from (10), (11) and (12) we obtain

$$T_2 + T_3 + T_4 = \sum_{k\in\sigma_2} (T_k^2 + T_k^3 + T_k^4).$$

Since $T_2 + T_3 + T_4 > 0$, we obtain that $\rho_2 = 1$, a contradiction. A similar argument shows that $T_3 > 0$.

Suppose now that $T_2 = 0$. From (10), $T^2 = (T_1^2, ..., T_n^2) = 0$, while from (13), (15), and (19), we obtain that

$$\alpha_k T_3 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^3 - \mu_k T_k^3 \geq 0, \qquad k \in \sigma_2.$$

¿From (16) we obtain

$$\alpha_k T_3 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^3 - \mu_k T_k^3 = 0, \qquad k \in \sigma_1.$$

Combining these two equations in matrix form, we obtain

$$\alpha T_3 + [P' - I] M T^3 \geq 0.$$

Multiplying both sides of the inequality by $CM^{-1}[I - P']^{-1}$, we obtain

$$\begin{pmatrix} \rho_1 - 1 \\ \rho_2 - 1 \end{pmatrix} T_3 + \begin{pmatrix} T_3 - \sum_{k\in\sigma_1} T_k^3 \\ T_3 - \sum_{k\in\sigma_2} T_k^3 \end{pmatrix} \geq 0.$$

Since from (11), $T_3 = \sum_{k\in\sigma_2} T_k^3$ and $T_3 > 0$, we obtain that $\rho_2 = 1$, a contradiction. By a similar argument $T_4 > 0$. □

We next prove that the condition of Theorem 1 is also necessary.

Theorem 2 (Necessary Conditions for stability) *If the linear program $LP[0]$ has a nonzero solution, then there exists a work-conserving policy under which the multiclass fluid network (α, μ, P, C) is unstable.*

Proof:

Let (T_j, T_k^j) be a nonzero solution of the linear program $LP[0]$. We will construct an initial condition $Q(0) \in R_1$ and a work-conserving policy, such that for some time $\tau > 0$,

$Q(\tau) = Q(0)$. It will follow that there exists a work-conserving policy under which the system never empties and therefore the fluid network is unstable.

Let

$$Q_k(0) = -(\alpha_k T_2 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^2 - \mu_k T_k^2), \qquad k \in \sigma_1$$

and

$$Q_k(0) = 0, \qquad k \in \sigma_2.$$

Constraint (18) guarantees that $Q(0) \geq 0$. We next show that $\sum_{k\in\sigma_1} Q_k(0) > 0$, i.e., $Q(0) \in R_1$. If $Q(0) = 0$, then, for all $k \in \sigma_1$

$$\alpha_k T_2 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^2 - \mu_k T_k^2 = 0$$

Moreover, from (14) for all $k \in \sigma_2$

$$\alpha_k T_2 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^2 - \mu_k T_k^2 \geq 0.$$

In matrix form, with $T^i = (T_1^i, \ldots, T_n^i)'$, the previous equations become

$$\alpha T_2 + [P' - I] M T^2 \geq 0.$$

Multiplying by $CM^{-1}[I - P']^{-1}$, we obtain

$$\begin{pmatrix} \rho_1 - 1 \\ \rho_2 - 1 \end{pmatrix} T_2 + \begin{pmatrix} T_2 - \sum_{k\in\sigma_1} T_k^2 \\ T_2 - \sum_{k\in\sigma_2} T_k^2 \end{pmatrix} \geq 0.$$

From (10), we have $T_2 = \sum_{k\in\sigma_1} T_k^2 = \sum_{k\in\sigma_2} T_k^2$. From Proposition 3, $T_2 > 0$, so $\rho_1, \rho_2 \geq 1$, a contradiction and therefore, $Q(0) \neq 0$.

We next construct the following allocation process for $k = 1, \ldots, n$:

$$T_k(t) = \begin{cases} \frac{t}{T_2} T_k^2 & t \in [0, T_2]; \\ T_k^2 + \frac{t-T_2}{T_3} T_k^3 & t \in (T_2, T_2 + T_3]; \\ T_k^2 + T_k^3 + \frac{t-T_2-T_3}{T_4} T_k^4 & t \in (T_2 + T_3, T_2 + T_3 + T_4]; \\ T_k^2 + T_k^3 + T_k^4 + \frac{t-T_2-T_3-T_4}{T_1} T_k^1 & t \in (T_2 + T_3 + T_4, T_2 + T_3 + T_4 + T_1]. \end{cases}$$

We show that the above allocation process is both feasible and work-conserving.

We first consider the first interval $[0, T_2]$. By the dynamics of the fluid network for this allocation process and starting from the initial condition given above we obtain

$$Q_k(T_2) = 0, \;\; k \in \sigma_1$$

$$Q_k(T_2) = \alpha_k T_2 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^2 - \mu_k T_k^2 \geq 0, \;\; k \in \sigma_2.$$

We next show that

$$\sum_{k \in \sigma_2} Q_k(T_2) > 0,$$

so $Q(T_2) \in R_2$. If we assume that

$$\alpha_k T_2 + \sum_{i=1}^{n} \mu_i p_{ik} T_i^2 - \mu_k T_k^2 = 0, \quad k \in \sigma_2,$$

then from (13) and (19) we obtain that

$$\alpha_k (T_3 + T_4) + \sum_{i=1}^{n} \mu_i p_{ik} (T_i^3 + T_i^4) - \mu_k (T_k^3 + T_k^4) = 0, \quad k \in \sigma_2.$$

Also from (16) and (17) we obtain that

$$\alpha_k (T_3 + T_4) + \sum_{i=1}^{n} \mu_i p_{ik} (T_i^3 + T_i^4) - \mu_k (T_k^3 + T_k^4) \geq 0, \quad k \in \sigma_1.$$

Written in matrix from, the two previous relations become

$$\alpha(T_3 + T_4) + [P' - I] M (T^3 + T^4) \geq 0.$$

Multiplying by $CM^{-1}[I - P']^{-1}$, we obtain

$$\begin{pmatrix} \rho_1 - 1 \\ \rho_2 - 1 \end{pmatrix} (T_3 + T_4) + \begin{pmatrix} T_3 + T_4 - \sum_{k \in \sigma_1} (T_k^3 + T_k^4) \\ T_3 + T_4 - \sum_{k \in \sigma_2} (T_k^3 + T_k^4) \end{pmatrix} \geq 0.$$

Since $T_3 + T_4 = \sum_{k \in \sigma_2} (T_k^3 + T_k^4)$ and $T_3 + T_4 > 0$, we obtain $\rho_2 \geq 1$, a contradiction. Therefore, $\sum_{k \in \sigma_2} Q_k(T_2) > 0$.

Since the allocation process is linear, we obtain:

$$\forall t \in [0, T_2], \qquad Q(t) \geq 0,$$

and

$$\forall t \in (0, T_2), \qquad Q(t) \in R_{12},$$

i.e., the allocation process is feasible. We next show that it is also work-conserving. From (10)

$$t = \sum_{k \in \sigma_1} \frac{t}{T_2} T_k^2 = \sum_{k \in \sigma_2} \frac{t}{T_2} T_k^2$$

or equivalently

$$\forall t \in [0, T_2]: \; U_1(t) = U_2(t) = U_1(0) = U_2(0) = 0,$$

and the process is indeed work-conserving.

In the interval $(T_2, T_2 + T_3]$, we prove similarly that for $k \in \sigma_2$ we have $Q_k(T_2 + T_3) \geq 0$ and $\sum_{k \in \sigma_2} Q_k(T_2 + T_3) > 0$. Therefore, $Q(T_2 + T_3) \in R_2$, and since $Q(T_2) \in R_2$, we obtain by linearity that

$$\forall t \in [T_2, T_2 + T_3], \; Q(t) \in R_2.$$

Work-conservation is shown similarly.

Similarly, we show that in the interval $t \in (T_2 + T_3, T_2 + T_3 + T_4]$, $Q(t) \in R_{12}$ and in the interval $t \in [T_2 + T_3 + T_4, T_2 + T_3 + T_4 + T_1]$, $Q(t) \in R_1$, while the process is work-conserving.

In addition, because of (19), $Q(T_1 + T_2 + T_3 + T_4) = Q(0)$. It follows that the fluid network never empties for this work-conserving feasible policy, and is unstable. □

The necessity proof has identified a particular way that an unstable work-conserving trajectory materializes, leading to some insight as to how instability may be reached. In particular, we have shown that if there exists an unstable trajectory, then there exists a periodic trajectory with a particular structure.

Combining Theorems 1 and 2 we obtain the main theorem of this section.

Theorem 3 *A two-station multiclass fluid network (α, μ, P, C) is stable for all work conserving policies if and only if the load condition $\rho < e$ holds and the linear program $LP[0]$ has zero as the only feasible solution.*

3.4 A special case

To illustrate the use (as well as the power) of Theorem 3 we prove that a two-station fluid network, in which one of the two stations has only one class, is stable provided that the load condition (4) is satisfied. This generalizes previous results obtained by Kumar [9] and Meyn and Down [8] for a three–class two–station network.

Theorem 4 *A fluid network satisfying the load condition $\rho < e$ with two stations and such that only one class is served by station 2 ($|\sigma_2| = 1$) is stable.*

Proof: We show that the corresponding linear program $LP[0]$ cannot have a nonzero solution. For the purposes of contradiction suppose that (T_j, T_k^j) is a nonzero solution to $LP[0]$. Let $\sigma_2 = \{l\}$. We distinguish two cases:

Case 1: $\alpha_l T_3 + \sum_{i=1}^n \mu_i p_{il} T_i^3 - \mu_l T_l^3 \geq 0$.

¿From (16):

$$\alpha_k T_3 + \sum_{i=1} \mu_i p_{ik} T_i^3 - \mu_k T_k^3 = 0, \ \forall k \in \sigma_1.$$

We combine the previous relations in matrix form as follows:

$$\alpha T_3 + [P' - I] M T^3 \geq 0.$$

We multiply both sides by $CM^{-1}[I - P']^{-1}$ to obtain:

$$\begin{pmatrix} \rho_1 - 1 \\ \rho_2 - 1 \end{pmatrix} T_3 + \begin{pmatrix} T_3 - \sum_{k \in \sigma_1} T_k^3 \\ T_3 - T_l^3 \end{pmatrix} \geq 0.$$

But from (11) we obtain $T_3 = T_l^3$ and from Proposition 3, we obtain $T_3 > 0$, leading to $\rho_2 = 1$, a contradiction.

Case 2 : $\alpha_l T_3 + \sum_{i=1}^n \mu_i p_{il} T_i^3 - \mu_l T_l^3 \leq 0$.

¿From (19), we obtain

$$\alpha_l (T_4 + T_1 + T_2) + \sum_{i=1}^n \mu_i p_{il} (T_i^4 + T_i^1 + T_i^2) - \mu_l (T_l^4 + T_l^1 + T_l^2) \geq 0.$$

Moreover, from (16) and (19) we obtain

$$\alpha_k (T_4 + T_1 + T_2) + \sum_{i=1}^n \mu_i p_{ik} (T_i^4 + T_i^1 + T_i^2) - \mu_k (T_k^4 + T_k^1 + T_k^2) = 0, \ k \in \sigma_1,$$

which, in matrix form, becomes

$$\alpha(T_4 + T_1 + T_2) + [P' - I]M(T^4 + T^1 + T^2) \geq 0.$$

Multiplying both sides by $CM^{-1}[I - P']^{-1}$ we obtain:

$$\begin{pmatrix} \rho_1 - 1 \\ \rho_2 - 1 \end{pmatrix} (T_4 + T_1 + T_2) + \begin{pmatrix} T_4 + T_1 + T_2 - \sum_{k \in \sigma_1} (T_k^4 + T_k^1 + T_k^2) \\ T_4 + T_1 + T_2 - (T_l^4 + T_l^1 + T_l^2) \end{pmatrix} \geq 0$$

¿From (9), (10), and (12)we obtain

$$T_4 + T_1 + T_2 = \sum_{k \in \sigma_1} (T_k^4 + T_k^1 + T_k^2),$$

and since $T_4 + T_1 + T_2 > 0$, then $\rho_1 = 1$, a contradiction. □

4 Sufficient stability conditions for a general multiclass fluid network

In this section we generalize the technique from the previous section to derive new sufficient conditions for stability of a general multiclass fluid network involving an arbitrary number J of stations.

Let us describe our approach in general terms. Recall that for any $S \subset \{1, \ldots, J\}$, we have defined R_S (cf. Section 2) as the set of all states Q for which all stations in S (resp., not in S) have a positive (resp., zero) number of customers. Consider an arbitrary work-conserving trajectory. As long as $Q(t) \neq 0$ this trajectory will be visiting the subspaces R_S, $S \subset \{1, \ldots, J\}$ in some arbitrary fashion. At any given point in time, the trajectory will be inside some R_S coming from some R_U and going to some R_V and we think of each possible triple (U, S, V) as a different type of behavior. Accordingly, we will partition the time axis into intervals such that during each interval the system exhibits the same type of behavior.

We now continue with a more formal development. Let T be the time that the system empties. (We let $T = \infty$ if the system never empties.) Then, it is easily shown (a formal proof is omitted) that there exists a countable collection of disjoint intervals (t_r, t'_r) such that:

(a) within each such interval, $Q(t)$ stays inside the same subspace R_S;

(b) these are maximal intervals with the property (a); formally, for every $\epsilon > 0$ there exist $t \in (t_r - \epsilon, t_r)$ and $t' \in (t'_r, t'_r + \epsilon)$ such that $Q(t) \notin R_S$ and $Q(t') \notin R_S$.

(c) these intervals together with their endpoints cover the entire interval $[0, T]$; in particular, the total length of these intervals is equal to T.

Let us focus on a typical such interval (t_r, t'_r) and let S be such that $Q(t) \in R_S$ for all $t \in (t_r, t'_r)$. We now need to define the subspace R_U that the state is coming from at the beginning of the interval. If $Q(t_r) \in R_U$ for some $U \neq S$, this is easy, and we say that the state is "coming" from R_U. If on the other hand, $Q(t_r) \in R_S$, we need to look at $Q(t)$ for times slightly less than t_r. Let us choose some U so that for every $\epsilon > 0$, $Q(t)$ visits R_U during the time interval $(t_r - \epsilon, t_r)$. (Note that the choice of U need not be unique.) We will again say that the state is "coming" from R_U.

Suppose that the state is coming from R_U. We consider in some more detail the two different possibilities.

(a) If $Q(t_r) \in R_S$, then every station $j \in S$ has a positive number of customers at time t_r. By continuity, this is also true just before t_r and we conclude that $U \supset S$.

(b) If $Q(t_r) \in R_U$, then every station $j \in U$ has a positive number of customers at time t_r. By continuity, this is also true just after t_r and we conclude that $U \subset S$.

The situation for the right endpoint t'_r of an interval is entirely similar. We can define some V such that $Q(t)$ is "going to" R_V. If $Q(t'_r) \in S$, we must have $V \supset S$; if $Q(t'_r) \in R_V$, we must have $V \subset S$.

Having determined for each interval where it is coming from and where it is going to, we can now assign to each interval a "type" (U, S, V). According to our earlier discussion, for any possible type, we must have either $U \subset S$ or $U \supset S$, and either $V \subset S$ or $V \supset S$. We refer to these as ***admissible*** types.

For any given trajectory and for any admissible type (U, S, V), we define the variable $T_S^{U,V}$ as the sum of the lengths of all intervals of type (U, S, V); intuitively, this is the total time the trajectory spends in R_S coming form R_U and going to R_V. Let $T_{S,k}^{U,V}$ be the total work allocated to class k during all intervals of type (U, S, V).

Note that the number of variables that we have introduced increases exponentially with the number of stations, because there are $2^J - 1$ choices for each subset U, S, V. A more precise estimate follows:

Proposition 4 *The total number of variables $T_S^{U,V}$ is*

$$\sum_{m=1}^{J} \binom{J}{m} [(2^m - 2)(2^m - 3) + (2^{J-m} - 1)(2^{J-m} - 2) + 2(2^m - 2)(2^{J-m} - 1)] = O(5^J).$$

Proof For $|S| = m$, there are the following cases:

a) $U \subset S$ and $V \subset S$ and therefore there are $(2^m - 2)(2^m - 3)$ choices for two nonempty subsets of S which are not S,

b) $S \subset U$ and $S \subset V$ and therefore there are $(2^{J-m} - 1)(2^{J-m} - 2)$ choices for two nonempty supersets of S which are not S,

c) $U \subset S \subset V$ or $U \subset S \subset V$ and therefore there are $2(2^m - 2)(2^{J-m} - 1)$ choices for one subset (which is not S and not empty) and one superset of S which is not S. □

Note that in total we have defined $O(n5^J)$ variables $T_{S,k}^{U,V}$.

Proceeding as in the two-station case, we first show the following upper bound on the duration of the strong busy period.

Proposition 5 *Consider a stable work-conserving policy $T(t)$ starting with initial condition $Q(0) \neq 0$. Let T be the smallest time such that $Q(T) = 0$. Then, T is bounded above by the optimal value in the following linear program to be called G[Q(0)]:*

$$\text{maximize} \sum_{(S,U,V)} T_S^{U,V}$$

subject to

$$\sum_{k \in \sigma_i} T_{S,k}^{U,V} = T_S^{U,V}, \quad i \in S, \tag{20}$$

$$\sum_{k \in \sigma_i} T_{S,k}^{U,V} \leq T_S^{U,V}, \quad i \notin S, \tag{21}$$

for $i \notin S,\ k \in \sigma_i$:

$$\alpha_k T_S^{U,V} + \sum_{i=1}^{n} \mu_i p_{ik} T_{S,i}^{U,V} - \mu_k T_{S,k}^{U,V} = 0, \tag{22}$$

$\forall i \in S \cap U^c \cap V^c,\ k \in \sigma_i$:

$$\alpha_k T_S^{U,V} + \sum_{i=1}^{n} \mu_i p_{ik} T_{S,i}^{U,V} - \mu_k T_{S,k}^{U,V} = 0, \tag{23}$$

$\forall i \in S \cap U^c \cap V,\ k \in \sigma_i$:

$$\alpha_k T_S^{U,V} + \sum_{i=1}^{n} \mu_i p_{ik} T_{S,i}^{U,V} - \mu_k T_{S,k}^{U,V} \geq 0, \tag{24}$$

$\forall i \in S \cap U \cap V^c,\ k \in \sigma_i$:

$$\alpha_k T_S^{U,V} + \sum_{i=1}^{n} \mu_i p_{ik} T_{S,i}^{U,V} - \mu_k T_{S,k}^{U,V} \leq 0, \tag{25}$$

$\forall k \in \{1, \ldots, n\}$:

$$\begin{aligned} \alpha_k \sum_{(S,U,V)} T_S^{U,V} + \sum_{i=1}^{n} \mu_i p_{ik} \sum_{(S,U,V)} T_{S,i}^{U,V} - \\ - \mu_k \sum_{(S,U,V)} T_{S,k}^{U,V} = -Q_k(0), \end{aligned} \tag{26}$$

$$T_{S,k}^{U,V} \geq 0,\ T_S^{U,V} \geq 0.$$

Proof: Consider an arbitrary stable work-conserving policy and define the variables $T_{S,k}^{U,V}$ and $T_S^{U,V}$ as in the discussion earlier in this section. Since the policy is stable, all of these are finite.

Equality (20) expresses work-conservation for all stations $i \in S$. Inequality (21) expresses the fact that the cumulative idleness for all stations $i \notin S$ should be nondecreasing.

Consider an interval (t_r, t_r') of type (U, S, V). We then have the following relations:

$$\sum_{k \in \sigma_i} Q_k(t_r) = 0,\ i \in U^c$$

$$\sum_{k \in \sigma_i} Q_k(t_r) \geq 0,\ i \in U$$

$$\sum_{k \in \sigma_i} Q_k(t_r') = 0,\ i \in V^c$$

$$\sum_{k\in\sigma_i} Q_k(t'_r) \geq 0, \; i \in V$$

Therefore, for $i \in S \cap U^c \cap V$, $Q_k(t'_r) - Q_k(t_r) \geq 0$. Writing the dynamics explicitly and summing over r we obtain (24). Relations (22), (23) and (25) follow an entirely similar logic. Finally, (26) expresses the fact that at time $T = \sum_{(U,S,V)} T^{U,V}_{S,k}$, the network empties. Maximizing this expression gives an upper bound on the time to empty the network. □

Remark: It is interesting to compare the constraints in G[Q(0)] with the constraints that we derived earlier for the two–station case. Note that G[Q(0] does not contain any constraints analogous to (22), (23), (24) and (25) for the case $i \in S \cap U \cap V$. It can be checked that in the context of LP[Q(0)], this corresponds to the fact that for $k \in \sigma_1$, we do not have any constraints involving the variables T_1 and T^1_k and, for that for $k \in \sigma_2$, we do not have any constraints involving the variables T_3 and T^3_k.

There is one minor discrepancy between the development in Section 3 and the development here, which is worth noting. In Section 3, we did not use different variables for the two interval types (R_1, R_{12}, R_1) and (R_{12}, R_1, R_{12}); in particular, any interval of the form $[t_{4m+1}, t_{4m+2}]$ consist in general of an interval of type (R_{12}, R_1, R_{12}) followed by a nonnegative number of intervals of type (R_1, R_{12}, R_1). Even though these are two different interval types, we only introduced in Section 3 a single set of variables, namely the variables T^1_k. There is a fundamental reason why the discrepancy between these two lines of development is immaterial: it can be easily shown that if a feasible work-conserving trajectory $Q(\cdot)$ has an interval (t_r, t'_r) of type (R_1, R_{12}, R_1), then there exists another feasible work-conserving trajectory $\hat{Q}(\cdot)$ with the following properties: (a) the two trajectories agree outside (t_r, t'_r); (b) $\hat{Q}(t) \in R_1$ for all $t \in (t_r, t'_r)$. By proceeding in this fashion, all intervals of type (R_1, R_{12}, R_1) can be eliminated, and this is done without affecting the stability properties of a trajectory.

The above outlined argument can be easily generalized to the multi–station case. In particular, it can be shown that we may ignore all types (U, S, U) with $S \supset U$. On the other hand, types (U, S, U) with $S \subset U$ cannot be eliminated.

We conclude this section by stating the sufficient conditions for stability.

Theorem 5 (Sufficient Conditions for stability) *Suppose that the load condition (4) holds. Consider the linear program G[0] obtained by setting Q(0)=0 in G[Q(0)]. If G[0] has zero as the only feasible solution, then the multiclass network* (α, μ, P, C) *is stable for all work-conserving policies.*

Proof: The argument is identical with the proof of Theorem 1. □

5 On the power of convex potential functions

It is well known that a multiclass fluid network is stable under all work conserving policies if and only if there exists some potential (Lyapunov) function which decreases along all possible trajectories. An example of such a potential function is the maximum (over all work conserving policies) of the time it takes for the system to empty. However, in order to prove that a system is stable, one needs to explicitly construct such a potential function, and this can be quite difficult. One possibility that has been investigated in the recent past is to restrict to a class of convex potential functions (quadratic or piecewise linear) and to use linear programming or other techniques in order to identify a suitable potential function within such a class (Kumar and Meyn [10], Botvich and Zamyatin [3], Dai and Weiss [7], Down and Meyn [8]).

The above approach begs the question of whether convex potential functions have the power to establish (sharp) necessary and sufficient conditions for stability. In other words, is it true that whenever a system is stable under all work conserving policies, there exists a convex Lyapunov function that testifies to this? In this section we show that this is not possible, i.e., the approach through monotone convex potential functions has limitations. In particular we find necessary and sufficient conditions for the existence of piecewise linear monotone convex potential function for multiclass fluid networks with two stations and provide an example of a stable network for which these conditions do not hold, and thus no monotone convex piecewise linear potential function exists. As any monotone convex potential function can be approximated arbitrarily closely by a piecewise linear monotone convex potential function, the limitation of the method follows.

Our general approach in this section is the following. We consider only two–station systems and focus on monotone piecewise linear convex potential functions (MPLCPF). We show that if a MPLCPF exists that establishes stability, then there also exists one that consists of only two linear pieces. We then find necessary and sufficient conditions for the existence of a MPLCPF with two pieces that establishes stability. As any convex potential function can be approximated by a MPLCPF, these conditions can be interpreted as necessary and sufficient conditions for the existence of any monotone convex potential function that establishes stability.

We start our development with a definition.

Definition 1 *A function $\Phi : R^n_+ \mapsto R_+$ is called a monotone piecewise linear convex potential function (MPLCPF) if:*

(a) There exist nonnegative vectors $L_1, \ldots, L_N$ such that

$$\Phi(x) = \max_{1 \le i \le N} L_i' x, \qquad \forall x \ge 0,$$

(b) for any feasible work-conserving trajectory $Q(t)$,

$$\frac{d}{dt}\Phi(Q(t)) \le -1,$$

whenever the derivative is defined.

It is easily checked that if a MPLCPF exists, then the fluid network is stable. We will now proceed to develop necessary and sufficient conditions for the existence of a MPLCPF for a two-station multiclass fluid network. Our first step is to prove that each one of the vectors L_i in the formula for Φ must satisfy a set of linear inequalities.

Proposition 6 *Suppose that $\Phi(x) = \max_{i=1,\ldots,N} L_i' x$ is a MPLCPF. Then,*

$$L_k'(\alpha + [P - I]M e_{ij}) \le -1, \qquad \forall i \in \sigma_1,\ j \in \sigma_2, \tag{27}$$

where e_{ij} is a vector whose ith and jth components are 1 and all other components are zero.

Proof

We assume, without any loss of generality, that for each $k \in \{1, \ldots, N\}$, there exists some $x_0 \geq 0$ such that

$$L'_k x_0 > \max_{i \neq k} L'_i x_0.$$

(Otherwise, we would have

$$\Phi(x) = \max_{i \neq k} L'_i x,$$

for all $x \geq 0$, and L_k could be ignored altogether from our subsequent development.) Furthermore, by possibly scaling x_0 and by using the continuity of linear functions, we can also assume that $x_0 > 0$. Using continuity once more, we also have

$$\Phi(y) = L'_k y, \tag{28}$$

for all y in a small enough neighborhood of x_0.

Let $U = (U_1, ..., U_n) \in R^n_+$ be any vector satisfying:

$$\sum_{i \in \sigma_1} U_i = \sum_{j \in \sigma_2} U_j = 1. \tag{29}$$

For small $t > 0$, we consider the allocation process $T(t) = Ut$. Let us show that for small t, this creates a feasible work-conserving trajectory $Q(t)$, starting from the initial state $Q(0) = x_0 > 0$. Since $x_0 > 0$, then for small $t > 0$ we must also have $Q(t) > 0$ and the trajectory is feasible. The trajectory is also work-conserving since the total utilization at each station is equal to 1. Since $\Phi(x)$ is a potential function, we have

$$\frac{d}{dt}\Phi(Q(t))|_{t=0} \leq -1.$$

For small $t > 0$ we have that $Q(t)$ is close to x_0 so by (28)

$$\nabla\Phi(Q(t))|_{t=0} = L_k.$$

But

$$\frac{d}{dt}Q(t)|_{t=0} = \alpha + [P - I]MU.$$

Therefore,

$$\frac{d}{dt}\Phi(Q(t)|_{t=0} = \nabla\Phi(Q(t))|_{t=0}\frac{d}{dt}Q(t)|_{t=0} = L_k'(\alpha + [P' - I]MU) \leq -1.$$

The latter inequality must be true for any U satisfying (29). In particular it should be satisfied for

$$U = e_{ij} = (0, 0, \dots, 1, 0, \dots, 0, 1, 0, \dots, 0)',$$

where the ones appear in positions i and j. Applying the previous inequality with $U = e_{ij}$ yields (27). □

The constraints (27) have been derived by considering allocations $T(t) = Ut$ corresponding to both stations being busy. We now derive other constraints by considering situations in which one of the stations may be underutilized while the other is busy. We start by defining two polyhedra P_1 and P_2. Intuitively, P_1 is the set of all allocation vectors under which station 1 is busy while station 2 is possibly underutilized and maintains its queues at a constant (zero) level. We let

$$P_1 = \{U = (U_1, ..., U_n)|\ 1 = \sum_{i\in\sigma_1} U_i \geq \sum_{j\in\sigma_2} U_j;\ \alpha_j + \sum_{i=1}^{n} \mu_i p_{ij} U_i - \mu_j U_j = 0,\ \forall j \in \sigma_2;\ U_j \geq 0.\} \tag{30}$$

$$P_2 = \{V = (V_1, ..., V_n)|\ 1 = \sum_{j\in\sigma_2} V_j \geq \sum_{i\in\sigma_1} V_i;\ \alpha_l + \sum_{i=1}^{n} \mu_i p_{il} V_i - \mu_l V_l = 0,\ \forall l \in \sigma_1;\ V_l \geq 0\} \tag{31}$$

Let $U^1, U^2, \dots, U^r$, and $V^1, V^2, \dots, V^s$, be the set of extreme points of the polyhedra P_1 and P_2 respectively.

Proposition 7 *(a) Suppose that there exists some $x_0 \in R_1$ such that $L_k'y = \Phi(y)$ for all $y \in R_1$ in some neighborhood of x_0. Then,*

$$L_k'(\alpha + [P - I]MU^i) \leq -1,\ i = 1, \dots, r. \tag{32}$$

(b) Suppose that there exists some $x_0 \in R_2$ such that $L_m'y = \Phi(y)$ for all $y \in R_2$ in some neighborhood of x_0. Then,

$$L_m'(\alpha + [P - I]MV^j) \leq -1,\ j = 1, \dots, s. \tag{33}$$

Proof: For any vector $U \in P_1$ consider the allocation process $T(t) = Ut$. It is easily checked that for small $t > 0$ and given the initial state $Q(0) = x_0 \in R_1$, this allocation creates a feasible work-conserving trajectory $Q(t)$. In particular, for $i \in \sigma_1$, we have $Q_i(t) > 0$, by continuity. Also, for $j \in \sigma_2$, the condition $\alpha_j + \sum_{i=1}^{n} \mu_i p_{ij} U_i - \mu_j U_j = 0$ in the definition of P_1 implies that $Q_j(t) = 0$. Finally, this allocation is clearly work-conserving because the total utilization of station 1 is 1.

Since we have a feasible work-conserving trajectory, we must have

$$\frac{d}{dt}\Phi(Q(t))|_{t=0} \leq -1.$$

For small t, we have that $Q(t)$ is close to x_0, so

$$\Phi(Q(t)) = L_k' Q(t).$$

Therefore,

$$L_k' \frac{d}{dt} Q(t)|_{t=0} = L_k'(\alpha + [P - I]MU) \leq -1,$$

for all $U \in P_1$. Applying the previous inequality for all the extreme points U^i of P_1 we obtain (32). A similar argument yields (33). □

We now define

$$\Lambda_1 = \{L \in \{L_1, \ldots, L_N\} \mid L \text{ satisfies } (32)\},$$

$$\Lambda_2 = \{L \in \{L_1, \ldots, L_N\} \mid L \text{ satisfies } (33)\}.$$

We now prove the following:

Proposition 8 *(a) The sets Λ_1 and Λ_2 are nonempty.*

(b) There holds

$$L_j' x \leq \max_{L \in \Lambda_1} L' x, \qquad \forall x \in R_1,\ j \in \Lambda_2, \tag{34}$$

$$L_j' x \leq \max_{L \in \Lambda_2} L' x, \qquad \forall x \in R_1,\ j \in \Lambda_1. \tag{35}$$

Proof: Consider R_1 which is a set of dimension $|\sigma_1|$. Consider some k and the set of points $x \in R_1$ for which $L_k' x = \Phi(x)$. This set is a polyhedron. Since the polyhedra corresponding to the different choices of k must cover the set R_1, it follows that at least one of these polyhedra contains a (relatively) open subset of R_1. With such a k, we have $L'_k y = \Phi(x)$ on some (relatively) open subset of R_1 and using the preceding proposition, we obtain that k satisfies (32) and Λ_1 is nonempty. The proof for Λ_2 is similar.

(b) Suppose, to derive a contradiction, that there exists some $j \in \Lambda_2$ and some $x \in R_1$ such that $L'_j x > \max_{L \in \Lambda_1} L'x$. In particular, we have $L_j \notin \Lambda_1$. Consequently, there exists an open set in R_1 on which the maximum in the definition of Φ is attained by some $L_m \notin \Lambda_1$. But this is a contradiction to the preceding proposition. □

In the proof to follow, we will also make use of the following result:

Proposition 9 *Let there be given some vectors $L, L_1, \ldots, L_p$. Then, the condition*

$$L'x \leq \max_{1 \leq i \leq p} L'_i x, \qquad \forall x \geq 0,$$

holds if and only if there exist $\theta_1, ..., \theta_p \geq 0$ such that

$$\sum_{1 \leq i \leq p} \theta_i = 1$$

and

$$L \leq \sum_{1 \leq i \leq p} \theta_i L_i,$$

where the last inequality is meant to hold componentwise.

Proof: This is a simple application of linear programming duality. □

We are now ready to state the first result of this section, which provides necessary conditions for the existence of MPLCPF.

Theorem 6 *Consider a two-station multiclass fluid network and suppose that $\Phi(x) = \max_{1 \leq k \leq N} L'_k x$ is a MPLCPF. Then, there exists a vector $M \in R^n_+$ satisfying (27) and (32) and a vector $N \in R^n_+$ satisfying (27) and (33), such that:*

$$M(i) \geq N(i), \ \forall i \in \sigma_1, \quad \text{and} \quad N(j) \geq M(j), \ \forall j \in \sigma_2. \tag{36}$$

Proof Let $\Lambda_1 = \{M_1, ..., M_t\}$ and $\Lambda_2 = \{N_1, ..., N_r\}$, i.e., $M_1, ..., M_t$ are the vectors L_k in the formula defining $\Phi(x)$, which satisfy (27) and (32), and $N_1, ..., N_r$ are the vectors L_m which satisfy (27) and (33).

We now use Proposition 8, as well as Proposition 9 to obtain an equivalent condition. We conclude that for each $k = 1, 2, ..., r$ we can find $\lambda_1^k, ..., \lambda_t^k \geq 0$, $\sum_{l=1}^t \lambda_l^k = 1$ such that:

$$N_k(i) \leq \sum_{l=1}^{t} \lambda_l M_l(i), \quad \forall i \in \sigma_1. \tag{37}$$

and for each $l = 1, 2, ..., t$, we can find $\theta_1^l, ..., \theta_r^l \geq 0$, $\sum_{k=1}^r \theta_k^l = 1$, such that:

$$M_l(j) \leq \sum_{k=1}^{r} \theta_r^l N_r(j), \quad \forall j \in \sigma_2. \tag{38}$$

Let $a = (a_1, ..., a_t)$ and $b = (b_1, ..., b_r)$ be two nonnegative vectors satisfying:

$$\sum_{l=1}^{t} a_l \geq 1, \ \sum_{k=1}^{r} b_k \geq 1.$$

Consider

$$M = \sum_{l=1}^{t} a_l M_l,$$

and

$$N = \sum_{k=1}^{r} b_k N_k.$$

Clearly, M satisfies (27) and (32) and N satisfies (27) and (33).

Multiplying all the inequalities in (37) by $b_1, b_2, ..., b_r$ and adding them, we obtain

$$N(i) \leq \sum_{k=1}^{r} \sum_{l=1}^{t} b_k \lambda_l^k M_l(i), \quad \forall i \in \sigma_1 \tag{39}$$

Similarly,

$$M(j) \leq \sum_{l=1}^{t} \sum_{k=1}^{r} a_l \theta_k^l N_k(j), \qquad \forall i \in \sigma_2. \tag{40}$$

We will prove that we may select $a_1, ..., a_t$ and $b_1, ..., b_r$ in such a way that for each $l = 1, 2, \ldots, t$:

$$\sum_{k=1}^{r} b_k \lambda_l^k = a_l, \tag{41}$$

and for each $k = 1, 2, \ldots, r$:

$$\sum_{l=1}^{t} a_l \theta_k^l = b_k. \tag{42}$$

In this case (39) and (40) are written as follows:

$$N(i) \leq \sum_{l=1}^{t} a_l M_l(i) = M(i), \qquad \forall i \in \sigma_1,$$

and

$$M(j) \leq \sum_{k=1}^{r} b_k N_k(j) = N(j), \qquad \forall j \in \sigma_2,$$

implying (36).

Conditions (41) and (42) are written as follows:

$$(a_1, ..., a_t, b_1, ..., b_r) = (a_1, ..., a_t, b_1, ..., b_r) \begin{pmatrix} 0 & \ldots & 0 & \theta_1^1 & \ldots & \theta_r^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \theta_1^t & \ldots & \theta_r^t \\ \lambda_1^1 & \ldots & \lambda_t^1 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^r & \ldots & \lambda_t^r & 0 & \ldots & 0 \end{pmatrix}$$

or in matrix form

$$(a, b) = (a, b)\, \Delta. \tag{43}$$

Since Δ is a stochastic matrix, it well known that there exists a nonnegative, non-zero solution (a, b) to (43). By multiplying this solution (a, b) by a sufficiently large number we can ensure that:

$$\sum_{l=1}^{t} a_l \geq 1, \ \sum_{k=1}^{r} b_k \geq 1.$$

The proof of the theorem is now complete. □

We next show that the conditions stated in the previous theorem are also sufficient for the existence of a MPLCPF for a multiclass fluid network with two stations.

Theorem 7 *Consider a two-station multiclass fluid network. Let $L_1, L_2 \in R_+^n$ be such that L_1 satisfies (27) and (32), L_2 satisfies (27) and (33), while both $M = L_1$ and $N = L_2$*

satisfy condition (36). Then the function

$$\Phi(x) = \max\{L_1 x, L_2 x\}$$

is a MPLCPF and the fluid network is stable for all work-conserving policies.

Proof Let $Q(t)$ be any feasible work-conserving trajectory in the fluid network. We will prove that for any $t_0 \geq 0$

$$\frac{d}{dt}\Phi(Q(t))|_{t=t_0} \leq -1. \tag{44}$$

wherever the derivative is defined.

Let $T(t) = (T_k(t))_{1\leq k\leq n}$ be the allocation process corresponding to the trajectory $Q(t)$. Suppose that $Q(t_0) \in R_1$ and that $Q(t)$ stays in R_1 for some time beyond t_0. Then, since the policy is work-conserving we obtain

$$\sum_{k\in\sigma_1} \frac{d}{dt}T_k(t)|_{t=t_0} = 1.$$

Since the second station has empty buffers, we obtain:

$$\alpha_k + \sum_{i=1}^{n} \mu p_{ik}\frac{d}{dt}T_i(t)|_{t=t_0} - \mu_k \frac{d}{dt}T_k(t)|_{t=t_0} = 0, \qquad \forall k \in \sigma_2. \tag{45}$$

Let $U_k = \frac{d}{dt}T_k(t)|_{t=t_0}$. Since the allocation process is nondecreasing, we have $U \in R^n_+$. Moreover, due to (45), $U \in P_1$, where P_1 is the polyhedron defined in (30). Now, since $Q(t_0) \in R_1$ then, by (36), we have

$$\Phi(Q(t_0)) = L_1' Q(t_0)$$

Therefore,

$$\frac{d}{dt}\Phi(Q(t)|_{t=t_0} = L_1'\frac{d}{dt}Q(t)\Big|_{t=t_0} = L_1'\Big(\alpha+[P'-I]M\frac{d}{dt}T(t)\Big|_{t=t_0}\Big) = L_1'\Big(\alpha+[P'-I]MU\Big) \leq -1$$

The last inequality holds since by assumption L_1 satisfies (32).

By a similar argument we show that (44) holds when $Q(t_0) \in R_2$ or $Q(t_0) \in R_{12}$, proving the theorem. □

We summarize the previous two theorems as follows.

Theorem 8 *There exists a piecewise linear potential function for a two-station fluid network if and only if the following linear program referred to as* $(LPOT)$ *on variables* $L_1, L_2 \in R^n_+$ *is feasible:*

$$L_1'(\alpha + [P - I]Me_{ij}) \leq -1, \ \forall i \in \sigma_1, \ j \in \sigma_2,$$

$$L_2'(\alpha + [P - I]Me_{ij}) \leq -1, \ \forall i \in \sigma_1, \ j \in \sigma_2,$$

where e_{ij} *is a vector with the ith and jth entry equal to 1 and all other entries equal to zero; in addition,*

$$L_1'\Big(\alpha + [P - I]MU^i\Big) \leq -1, \ i = 1, \ldots, r,$$

where $U^1, U^2, \ldots, U^r$*, is the set of extreme points of the polyhedron* P_1 *defined in (30);*

$$L_2'\Big(\alpha + [P - I]MV^j\Big) \leq -1, \ j = 1, \ldots, s,$$

where $V^1, \ldots, V^s$ *is the set of extreme points of the polyhedron* P_2 *defined in (31);*

$$L_1(i) \geq L_2(i), \quad \forall i \in \sigma_1,$$

$$L_2(j) \geq L_1(j), \quad \forall j \in \sigma_2,$$

$$L_1, L_2 \geq 0.$$

Remarks:

1) The previous theorem can be used as a sufficient test for stability as follows. If $(LPOT)$ is feasible, then a potential function exists and the network is stable. If not, we can only conclude that a MPLCPF does not exist; no conclusion can be reached as to whether the network is stable or not. In comparison with the earlier work of Down and Meyn [8] and Dai and Weiss [7], the linear program $(LPOT)$ is the best possible result based on MPLCPFs, since it is guaranteed to discover a MPLCPF whenever one exists. It is thus sharper than earlier results.

2) The previous theorem can be easily generalized to the case of more than two stations. However, the necessary and sufficient conditions for the existence of a MPLCPF amount to a nonlinear programming problem; the reason is that the generalization of the condition (36) turns out to be nonlinear. Moreover, we expect that the linear program of Theorem 5 gives the sharp stability conditions.

5.1 Monotone convex potential functions are not necessary for stability

In the previous subsection we have established necessary and sufficient conditions for a MPLCPF to exist in a two-station fluid network. A natural question is how these conditions are related with the results of Section 3 (necessary and sufficient conditions for stability).

Consider the following example of a two-station fluid network (see Figure 2).

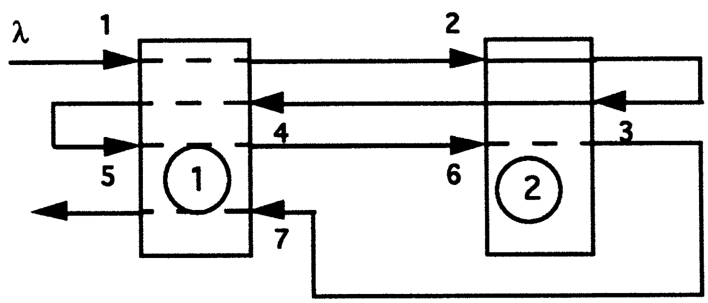

Figure 2: An example of a two-station fluid network.

There are 7 classes with rates $\mu_1 = 10$, $\mu_2 = 2.5$, $\mu_3 = 20$, $\mu_4 = 2$, $\mu_5 = 4$, $\mu_6 = 3$ and $\mu_7 = 11$. The external arrival rate to class 1 is $\lambda = 0.805$. Then $\sigma_1 = \{1, 4, 5, 7\}$ and $\sigma_2 = \{2, 3, 6\}$.

The traffic intensities are $\rho_1 = 0.7527$ and $\rho_2 = 0.6266$. The linear programming $LP[0]$ of Section 3 finds that 0 is the only feasible solution, which means that the system is stable.

The linear program $(LPOT)$ for the same data is infeasible, which implies that there is no MPLCPF, even though the system is stable. We note that for $\lambda = 0.804$, $LP[0]$ has 0 as the only feasible solution and $(LPOT)$ is feasible implying that a MPLCPF exists. Moreover, for $\lambda = 0.806$, $LP[0]$ has a nonzero solution and therefore the system is unstable, while $(LPOT)$ is infeasible. In other words, for this example, $(LPOT)$ correctly identifies stability for all $\lambda < 0.804$, while it is inconclusive for $\lambda = 0.805$, (even though the system is stable) and $\lambda = 0.806$, (while the system is unstable).

6 Conclusions

For two-station multiclass fluid network we have established

(a) necessary and sufficient conditions for stability of all work-conserving policies,

(b) necessary and sufficient conditions for existence of a monotone convex, piecewise linear potential function,

(c) an example of a stable system for which no MPLCPF exists, which implies that the convex potential function method has inherent limitations.

For networks with more than two stations we have established sufficient conditions for stability and we believe that these conditions are also necessary.

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